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Some 6-*j* symbols for symplectic and orthogonal groups by Cerkaski's method

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Abstract. The method devised by Cerkaski has been used for the group Sp($2n$) to find some 6-j symbols for which a single multiplicity index is required. This approach possesses some advantages over that based on Racah's techniques for SO(3): the formulae for the 6-j symbols are comparatively simple, and they automatically exhibit the symmetries of Jucys. Elementary means are used to show that the eigenvalues of Casimir's operator C_2 for Sp($2n$) are identical to those of C_2 for O($2d$) if we take $d=-n$ and replace the irreps of Sp($2n$) by their transposed counterparts for O($2d$), where d is integral or half-integral. This enables the formulae for Sp($2n$) to be used to generate expressions for some 6-j symbols for the orthogonal groups O($2d$) and SO($2d$). Illustrations are provided by SO($2l+1$) and O(4).

1. Introduction

In an analysis of considerable ingenuity, Cerkaski (1987) has shown how a class of 6-j symbols with one multiplicity index can be calculated for the groups Sp($2n$), SO($2n$), and SO($2n+1$). The essence of his approach is to use the generalized Biedenharn–Elliott identity as an equation determining the acceptable eigenfunctions associated with the diagonalization of a secular matrix. The roots of the secular equation and the diagonal elements of the secular matrix are known in terms of various eigenvalues of the second-order Casimir operator C_2. Many of the elements of the secular matrix can be chosen to be zero, a structural decision that forces the solution of the multiplicity problem to take a particular form. It also happens that this information is enough to fix (to within a phase) the non-vanishing off-diagonal elements. This, in turn, allows the eigenfunctions to be found and consequently a set of 6-j symbols.

Cerkaski illustrated the general formalism by giving the key algebraic quantities for a special case. However, he did not put the parts together to give explicit formulae for any 6-j symbols. It occurred to us that it would be interesting to do so. How well, we may ask, does Cerkaski's approach work in practice? In the absence of a sound strategy, clumsy algebraic expressions can easily arise when multiplicity separations are being made. We usually want to avoid quadratic forms that cannot be factored into two linear parts with rational coefficients, since such forms tend to produce high prime numbers in a numerical evaluation. Butler (1981, p 76) has given the absence of high prime numbers as a criterion for preferring one multiplicity separation over another for the icosahedral group, a point of view that we feel should be given considerable weight. Any underlying structure stands a better chance of being recognized when the mathematical verbiage is reduced to a minimum.

There is another reason for examining Cerkaski's approach. Cvitanović and Kennedy (1982) have stated that the *n-j* symbols for Sp($2n$) and SO($2n$) go into each other if the dimension $2n$ is replaced by $-2n$ and if symmetrization and antisymmetrization (as determined by the Young tableaux from which the irreps are derived) are interchanged. Their derivation uses diagrammatic techniques that have been developed in previous articles (Kennedy 1981, 1982) and which the casual reader must find disconcertingly ineffable at first sight. How, it might be asked, should SO($2n+1$) be treated? What happens if the symmetry interchange leads to a pair of irreducible representations (irreps) of SO($2n$) (which would occur if their final weights are non-zero) instead of the single encompassing irrep of O($2n$)? Cerkaski devised his method for Sp($2n$), SO($2n$) and SO($2n+1$) separately, and it would be of considerable interest to show how the use of negative dimensions fits into the picture. In working this out in section 6 below, we take the opportunity to treat the reciprocity between the orthogonal and symplectic groups in algebraic rather than diagrammatic terms, our aim being to make the analysis more accessible to the general reader.

2. The method applied

It was at once clear to us that Cerkaski's method could be used to find the 6-*j* symbols

$$\begin{Bmatrix} \langle\sigma 1\rangle & \langle 1\rangle & \langle\lambda\rangle \\ \langle 1\rangle & \langle\sigma 1\rangle & \langle\mu\rangle \end{Bmatrix}_r \tag{1}$$

for the group Sp($2n$). The multiplicity index r is required to separate each of the two duplications that occur when $\langle\mu\rangle=\langle 11\rangle$ and $\langle 2\rangle$. Here and in (1) we denote irreps of Sp($2n$) by their highest weight (omitting any zeros from the angular brackets). The results of the calculation are set out in table 1. The two values of r are denoted by a and b and attached as a subscript to $\langle\mu\rangle$. To avoid factorial functions in the tabulation, the 6-*j* symbols (1) are converted to the U coefficients of Jahn (1951) by multiplication by $[\mathrm{Dim}\langle\lambda\rangle\,\mathrm{Dim}\langle\mu\rangle]^{1/2}$.

A set of special cases for $n=3$ and $\sigma=2$ has already been presented in a preliminary report on Cerkaski's method (Judd 1987). We should also mention here that the isomorphism Sp(4) = SO(5) relates the entries of table 1 to the 6-*j* symbols of the group SO(5) when we set $n=2$. Since the weight space for Sp(4) is two-dimensional, all entries for $\langle\lambda\rangle=\langle\sigma 11\rangle$ can be ignored. Furthermore, $\langle\sigma 1\rangle\times\langle\sigma 1\rangle$ contains $\langle 11\rangle$ only once, so at first sight the two sets of symbols labelled by $\langle 11\rangle_\mathrm{a}$ and $\langle 11\rangle_\mathrm{b}$ describe a multiplicity separation where none exists. However, all of the entries corresponding to $\langle 11\rangle_\mathrm{b}$, with the exception of the irrelevant case for which $\langle\lambda\rangle=\langle\sigma 11\rangle$, contain $n-2$ as a factor and thus disappear. The remaining entries must be identical to the SO(5) U-coefficients

$$U\begin{pmatrix} (w_1 w_2) & (\frac{1}{2}\frac{1}{2}) & W \\ (\frac{1}{2}\frac{1}{2}) & (w_1 w_2) & W' \end{pmatrix}$$

where $w_1=\frac{1}{2}(\sigma+1)$ and $w_2=\frac{1}{2}(\sigma-1)$.

3. An alternative approach

The 6-*j* symbols (1) were of interest to us for several reasons. We had gained some experience in calculating multiplicity-free 6-*j* symbols for SO($2l+1$) (Judd *et al* 1986)

Table 1. Values of $U\begin{pmatrix} \langle\sigma 1\rangle & \langle 1\rangle & \langle\lambda\rangle \\ \langle 1\rangle & \langle\sigma 1\rangle & \langle\mu\rangle \end{pmatrix}_r$ for Sp($2n$).

	$\langle\lambda\rangle$				
$\langle\mu\rangle$	$\langle\sigma-1,1\rangle$	$\langle\sigma\rangle$	$\langle\sigma 11\rangle$	$\langle\sigma 2\rangle$	$\langle\sigma+1,1\rangle$
$\langle 0\rangle$	$-\left(\frac{(\sigma-1)(\sigma+1)(\sigma+2n-1)}{2n\sigma(\sigma+2n-2)(\sigma+2n)}\right)^{1/2}$	$\left(\frac{(\sigma+1)(\sigma+2n-1)}{2n\sigma(2n-2)(\sigma+2n)}\right)^{1/2}$	$\left(\frac{(2n-1)(2n-4)(\sigma+1)(\sigma+2n-1)}{4n(2n-2)(\sigma+2)(\sigma+2n-2)}\right)^{1/2}$	$-\left(\frac{(2n-1)(\sigma-1)(\sigma+2n+1)}{4n\sigma(\sigma+2n)}\right)^{1/2}$	$\left(\frac{(\sigma+1)(\sigma+2n-1)(\sigma+2n+1)}{2n\sigma(\sigma+2)(\sigma+2n)}\right)^{1/2}$
$\langle 11\rangle_a$	$-\left(\frac{(2n-1)(2\sigma+2n-2)^2(\sigma+2n+1)}{4n\sigma(2n+2)(\sigma+2n-2)(\sigma+2n)}\right)^{1/2}$	$-\left(\frac{(2n-2)(2n-1)(\sigma-1)(\sigma+2n+1)}{4n\sigma(2n+2)(\sigma+2n)}\right)^{1/2}$	$\left(\frac{(2n-2)(2n-4)(\sigma-1)(\sigma+2n+1)}{8n(2n+2)(\sigma+2)(\sigma+2n-2)}\right)^{1/2}$	$\left(\frac{(2n+2)(\sigma+1)(\sigma+2n-1)}{8n\sigma(\sigma+2n)}\right)^{1/2}$	$\left(\frac{(2n-1)(2\sigma+2n+2)^2(\sigma-1)}{4n\sigma(2n+2)(\sigma+2)(\sigma+2n)}\right)^{1/2}$
$\langle 11\rangle_b$	$-\left(\frac{(2n-4)(\sigma-1)(\sigma+2n)}{2\sigma(2n+2)(\sigma+2n-2)}\right)^{1/2}$	$\left(\frac{(2\sigma+2n)^2(2n-4)}{2\sigma(2n-2)(2n+2)(\sigma+2n)}\right)^{1/2}$	$-\left(\frac{(2\sigma+2n)^2(2n-1)}{(2n-2)(2n+2)(\sigma+2)(\sigma+2n-2)}\right)^{1/2}$	0	$\left(\frac{\sigma(2n-4)(\sigma+2n+1)}{2(2n+2)(\sigma+2)(\sigma+2n)}\right)^{1/2}$
$\langle 2\rangle_a$	$\left(\frac{(\sigma-1)(\sigma+2n)}{2\sigma(\sigma+2n-2)}\right)^{1/2}$	$-\left(\frac{4n^2}{2\sigma(2n-2)(\sigma+2n)}\right)^{1/2}$	$-\left(\frac{(2n-1)(2n-4)}{(2n-2)(\sigma+2)(\sigma+2n-2)}\right)^{1/2}$	0	$\left(\frac{\sigma(\sigma+2n+1)}{2(\sigma+2)(\sigma+2n)}\right)^{1/2}$
$\langle 2\rangle_b$	$\left(\frac{(2n-1)(\sigma+2n+1)}{2\sigma(\sigma+2n-2)(\sigma+2n)}\right)^{1/2}$	$\left(\frac{(2n-1)(\sigma-1)(\sigma+2n+1)}{2\sigma(2n-2)(\sigma+2n)}\right)^{1/2}$	$\left(\frac{(2n-4)(\sigma-1)(\sigma+2n+1)}{4(2n-2)(\sigma+2)(\sigma+2n-2)}\right)^{1/2}$	$\left(\frac{(\sigma+1)(\sigma+2n-1)}{4\sigma(\sigma+2n)}\right)^{1/2}$	$\left(\frac{(2n-1)(\sigma-1)}{2\sigma(\sigma+2)(\sigma+2n)}\right)^{1/2}$

and for G_2 (Judd 1987), and we wanted to study how a multiplicity index could be handled. We also wanted to extend the range of our techniques by picking a different class of Lie groups. The method we were using differed from Cerkaski's. Our central idea sprang from the original work of Racah (1942) in which he introduced the W coefficient (an unsymmetrized 6-j symbol) to calculate the matrix elements of scalar products of the type $\boldsymbol{T}_{\mathrm{A}}^{(k)} \cdot \boldsymbol{T}_{\mathrm{B}}^{(k)}$ for the atomic configuration $l_{\mathrm{A}} l_{\mathrm{B}}$, where the rank k of the tensors specifies an irrep of SO(3). To work out the 6-j symbols (1) by an extension of our earlier method, the atomic configuration $l_{\mathrm{A}} l_{\mathrm{B}}$ must be replaced by $j^{\sigma} j' j''$, where j, j' and j'' denote three inequivalent bosons, each with identical angular momentum j. We can introduce Sp$(2j+1)$ to describe the transformation properties of these states; therefore $2j+1=2n$. The scalar product is replaced by

$$S_{\mathrm{AB}} \boldsymbol{T}_{\mathrm{A}}^{\langle\mu\rangle} \cdot \boldsymbol{T}_{\mathrm{B}}^{\langle\mu\rangle} + S_{\mathrm{AC}} \boldsymbol{T}_{\mathrm{A}}^{\langle\mu\rangle} \cdot \boldsymbol{T}_{\mathrm{C}}^{\langle\mu\rangle} + S_{\mathrm{BC}} \boldsymbol{T}_{\mathrm{B}}^{\langle\mu\rangle} \cdot \boldsymbol{T}_{\mathrm{C}}^{\langle\mu\rangle} \tag{2}$$

where the suffices A, B and C refer to the three parts j^{σ}, j' and j''. The coefficients S_{PQ} determine the relative strengths of the scalar products and are at our disposal. The actual construction of the states of $j^{\sigma} j' j''$ belonging to a given coupling $(\langle\sigma 1\rangle\langle 1\rangle)\langle\lambda\rangle$ can be largely avoided by a judicious use of the diagonal sum rule, and the 6-j symbol under study can be related to the three strengths S_{PQ} appearing in the linear combination (2). If $\langle\mu\rangle$ occurred just once in the decomposition of the Kronecker square $\langle\sigma 1\rangle^2$, all three operators in (2) would yield matrix elements proportional to one another. This is not so for $\langle\mu\rangle = \langle 11\rangle$ or $\langle 2\rangle$. In these cases the matrix elements depend on the relative strengths of the coefficients S_{AB}, S_{AC} and S_{BC}, though the first two of these always appear in the combination $S_{\mathrm{AB}} + S_{\mathrm{AC}}$. This is a consequence of the symmetry between B and C for the configuration $j^{\sigma} j' j''$. We can thus generate one string of 6-j symbols for which r corresponds to the coefficients of $S_{\mathrm{AB}} + S_{\mathrm{BC}}$, and a second string for which r $(=r'$, say) corresponds to the coefficients of S_{BC}.

It was at once found that the multiplicity labels a and b of table 1 do not coincide with r and r'. This is not surprising, of course, since we cannot expect two distinct methods to lead to identical multiplicity separations. More importantly, we cannot be sure that the labels r and r' correspond to orthogonal rows of the U matrix. Orthogonality is guaranteed by Cerkaski's method because his results stem from the diagonalization of a real Hermitian matrix. The method of the paragraph above leads to orthogonal rows provided a suitable choice is made for the strengths S_{PQ}. For example, we can generate Cerkaski's results for $\langle\mu\rangle = \langle 2\rangle_{\mathrm{a}}$ by taking an interaction for which $S_{\mathrm{AB}} = 0$ and $S_{\mathrm{AC}} = S_{\mathrm{BC}}$. In this case the operator (2) possesses eigenvalues for the state $(\langle\sigma 1\rangle\langle 1\rangle)\langle\lambda\rangle$ proportional to the combination

$$G\langle\lambda\rangle - G\langle 1\rangle - G\langle\sigma 1\rangle$$

where $G\langle\lambda\rangle$ is the eigenvalue of Casimir's operator C_2 for Sp$(2n)$.

4. Jucys' symmetries

There is a second difference between the two methods of resolving the multiplicity ambiguities. The analogue of the substitutions $j \rightarrow -j-1$ for SO(3), which leave invariant the characteristic quadratic form $j(j+1)$, is

$$\sigma \rightarrow -\sigma - 2n \tag{3}$$

for $\langle\sigma 1\rangle$, since $G\langle\sigma 1\rangle$ is proportional to the product $(\sigma+2n-1)(\sigma+1)$ (Wybourne 1974). Substitutions that leave the eigenvalues of Casimir's operators invariant have been studied extensively by Jucys (1970) and his colleagues (Jucys and Savukynas 1973, Ališauskas 1987). Other $G\langle\lambda\rangle$ are not necessarily invariant under the substitution (3); however, we can always find a representation $\langle\lambda'\rangle$ for which $G\langle\lambda\rangle\to G\langle\lambda'\rangle$. For the irreps appearing in table 1, for example, it is easy (with the aid of equation (15.6) of Wybourne (1974)) to verify the correspondences $\langle\sigma-1, 1\rangle\to\langle\sigma+1, 1\rangle$, $\langle\sigma 0\rangle\to\langle\sigma 0\rangle$, $\langle\sigma 11\rangle\to\langle\sigma 11\rangle$, and $\langle\sigma 2\rangle\to\langle\sigma 2\rangle$. Since Cerkaski's method depends crucially on using Casimir's operator C_2, we would expect the entries of table 1 to exhibit Jucys' symmetries; and, indeed, the first and fifth columns, which are associated through the interchange $\langle\sigma-1, 1\rangle\leftrightarrow\langle\sigma+1, 1\rangle$, are reciprocally connected (to within a phase factor) by the substitutions (3). The use of the operator (2), on the other hand, does not impose Jucys' symmetry on the calculation. While this gives us greater freedom in making the multiplicity separation, there is no particular advantage in doing so. In fact, we normally need guidance of some kind to avoid ungainly algebraic combinations.

5. Negative dimensions

The close connection between the orthogonal and symplectic groups can be seen by comparing the branching rules for the decomposition of an irrep $[\lambda]$ of $U(N)$ (Wybourne 1970, pp 39 and 42). We have only to interpret $[\lambda]$ as a Young tableau and reflect in the diagonal all manipulations with the cells (that is, transpose all operations by interchanging rows and columns) to pass from one scheme to the other. Every irrep of the full orthogonal group corresponds to an irrep of the symplectic group. The dimensions of these two groups need not be the same; but, should the tableau reflection of the irrep of one group lead to more rows than the dimension of the weight space of the other, the modification rules of Murnaghan (1938) have to be brought into play. (See also Wybourne (1970), pp 42-5.) For groups whose dimensions N are sufficiently large, the paired irreps $(w_1w_2\ldots)$ and $\langle\sigma_1\sigma_2\ldots\rangle$ end with a string of zeros and no adjustments are necessary. Since $O(N)$ possesses an l-dimensional weight space for both $N=2l$ and $N=2l+1$, we can cope with orthogonal groups in an even and an odd number of dimensions at the same time: their irreps are both of the type $(w_1w_2\ldots w_l)$. The reflection operation takes $\langle\sigma_1\sigma_2\ldots\sigma_n\rangle$ of $Sp(2n)$ into W of $O(N)$, where

$$W=(n^{\sigma_n}(n-1)^{\sigma_{n-1}-\sigma_n}(n-2)^{\sigma_{n-2}-\sigma_{n-1}}\ldots 1^{\sigma_1-\sigma_2}0^{l-\sigma_1}). \tag{4}$$

The powers of the weights indicate the number of occurrences: that is, $w_1=n$, $w_2=n,\ldots, w_{\sigma_n}=n$, $w_{\sigma_{n-1}}=n-1$, etc. The easiest way to see how the structure of W comes about is to take the tableau $[\sigma_1\sigma_2\ldots\sigma_n]$ and read off the number of cells in successive columns. There is no difficulty in doing this when $l\geqslant\sigma_1$, since we have enough dimensions in the weight space of $O(N)$ to accommodate all the rows of the reflected tableau. When this condition is not satisfied, the modification rules must be introduced. We return to this point in section 7.

The relevance of negative dimensions becomes apparent when the eigenvalues of Casimir's operator C_2 are calculated. From equations (15.5)-(15.7) of Wybourne (1974) we get

$$G\langle\sigma_1\sigma_2\ldots\sigma_n\rangle=\tfrac{1}{4}(n+1)^{-1}\sum_{i=1}^{n}[(\sigma_i+n-i+1)^2-(n-1+i)^2] \tag{5}$$

for Sp($2n$) and

$$G(w_1 w_2 \ldots w_l) = \tfrac{1}{4}(d-1)^{-1} \sum_{i=1}^{l} [(w_i + d - i)^2 - (d-i)^2] \tag{6}$$

for O($2d$), where d can be integral (in which case $d = l$) or half-integral (in which case $d = l + \frac{1}{2}$). Equation (6) is equally valid for SO($2d$), but we prefer to work with the full orthogonal group to avoid the doubling of the irreps that occurs when $d = l$ and $w_l \neq 0$. To evaluate $G(W)$, we first define $T(q)$ by means of the equation

$$T_k(q) = (d-l)^k + (d-l+1)^k + (d-l+2)^k + \ldots + (q)^k. \tag{7}$$

Using equation (6), we see that the first σ_n weights (all equal to n) of W produce a contribution of $T_2(n+d-1) - T_2(n+d-1-\sigma_n)$ to the summation in equation (6); the next $\sigma_{n-1} - \sigma_n$ weights (all equal to $n-1$) of W produce a contribution of $T_2(n+d-2-\sigma_n) - T_2(n+d-2-\sigma_{n-1})$ to the summation; and so on until, for the $l - \sigma_1$ weights all equal to 0, we get the contribution $T_2(d - \sigma_1 - 1)$. When we add all these contributions we get

$$T_2(n+d-1) - (n+d-1-\sigma_n)^2 - (n+d-2-\sigma_{n-1})^2 \ldots - (d-\sigma_1)^2. \tag{8}$$

From this we must subtract the second term in the square brackets in equation (6); that is, we must subtract $T_2(d-1)$. This term can be combined with the term $T_2(n+d-1)$ in the expression (8), with the result that we have

$$G(W) = -\tfrac{1}{4}(d-1)^{-1} \sum_{i=1}^{l} [(d - \sigma_i + i - 1)^2 - (d+i-1)^2]. \tag{9}$$

This becomes identical to $G\langle \sigma_1 \sigma_2 \ldots \sigma_n \rangle$ given in equation (5) if we replace d by $-n$. That is, Casimir's operator C_2 for Sp($2n$) possesses eigenvalues that are identical to those of C_2 for O($2d$) when we reflect the tableaux and replace n by $-d$.

A referee has kindly pointed out to us that it would be useful to mention that similar correspondences must occur for SU(n) and SU(d). Indeed, Haase and Butler (1985) have noticed that some elementary 6-j symbols for the unitary groups go into each other when we reflect the tableaux and replace n by $-n$. The interested reader is referred to other articles by members of the Christchurch school (Bickerstaff and Wybourne 1981, Bickerstaff 1984, Haase and Butler 1984, Haase and Dirl 1986).

6. Generalizations

The occurrence of negative dimensions has been explored in terms of Grassmann variables by Dunne (1989). He showed that the pth-order Casimir operators C_p for SO($2n$) and Sp($2n$) possess eigenvalues for the totally symmetric irreps (either $(r0^{n-1})$ or $\langle r0^{n-1}\rangle$) that go over into $(-1)^p$ times the eigenvalues of the totally antisymmetric irreps of the other group (either $\langle 1^r 0^{n-r}\rangle$ or $(1^r 0^{n-r})$) if n is replaced by $-n$. The phase differs trivially from ours because of his omission of the factors preceding the summation signs in equations (5) and (6). Equation (9) shows that, for $p = 2$, this result can be generalized from tableaux possessing single columns and single rows to any pair related by the reflection operation. Mkrtchyan (1981, note vi) has stated without proof that it is true for any p. We can easily adapt our derivation to confirm this by noting that Popov and Perelomov (1968, equation (41)) have expressed the eigenvalues of C_p in terms of the sums of products of the type appearing in square brackets in equations

(5) and (6), the only difference being that various powers k appear instead of the squares. However, the analysis leading from equation (6) to equation (9) is unchanged if every T_2 replaced by the corresponding T_k. Furthermore, the coefficients in the expansion used by Popov and Perelomov (1968, table II) go into each other when the substitution $d \to -n$ is made, as can be verified in general by using their generating function for the C_p. The upshot of all of this is that any multiplicity separation resting on the diagonalization of any of the C_p must carry over from O($2n$) to Sp($2d$) on making the substitution $d \to -n$.

7. Special cases

Since Cerkaski's method depends on the properties of C_2, we know that we can generate 6-j symbols for O($2d$) by taking the formulae for the 6-j symbols of Sp($2n$) and replacing n by $-d$. At the same time, the representation labels must be changed by interchanging the rows and columns of the corresponding tableaux. This is easily done for the entries of table 1:

$$\begin{array}{ll} \langle\sigma-1, 1\rangle \to (21^{\sigma-2}) & \langle\sigma\rangle \to (1^{\sigma}) \\ \langle\sigma 11\rangle \to (31^{\sigma-1}) & \langle\sigma 2\rangle \to (221^{\sigma-2}) \\ \langle\sigma+1, 1\rangle \to (21^{\sigma}) & \langle 0\rangle \to (0) \\ \langle 11\rangle \to (2) & \langle 2\rangle \to (11). \end{array} \tag{10}$$

The actual substitution $n \to -d$ requires some care because a naive application to an expression like $(n^2)^{1/2}$ would yield $+d$ rather than $-d$. To ensure that the new U coefficients form orthonormal matrices, the number of sign reversals under a square-root sign in the numerator (x, say) and in the denominator (y, say) should be counted and the new U coefficient multiplied by $(-1)^{(x-y)/2}$. This factor keeps track of the factors i that are produced by each sign reversal.

As an example, we construct the U coefficients

$$U\begin{pmatrix} (20\ldots0) & (10\ldots0) & (\lambda) \\ (10\ldots0) & (20\ldots0) & (\mu) \end{pmatrix} \tag{11}$$

for SO($2l+1$). We set $\sigma = 1$ and $2n = -2l-1$ in table 1. The irreps $\langle\sigma-1, 1\rangle$ and $\langle\sigma 2\rangle$ no longer specify highest weights and are unacceptable. This removes two columns from the table. Of the entries remaining in the five rows, all those in the rows $\langle 11\rangle_a$ and $\langle 2\rangle_b$ contain $(\sigma-1)$ as a factor and disappear. We are left with a three-by-three matrix. The factor $(-1)^{(x-y)/2}$ produces only one sign reversal, namely that for the first entry. The final matrix is given in table 2. The entries agree (to within a phase) with those given a few years ago (Judd *et al* 1986, table 1).

Sometimes the modification rules have to be used when transforming Sp($2n$) to a group O($2d$) with small d. If, for example, $2d = 7$ and $\sigma = 4$, the substitutions (10) show that the irreps (211), (1111), (3111), (2211) and (21111) of O(7) appear. Since the weight space of O(7) is three-dimensional, only the first irrep is immediately acceptable. The others convert to (111), (311), (221) and (210), respectively. We can check that the modification rules provide the right collection of irreps by referring to table D-4 of Wybourne (1970) for the reduction of the Kronecker product (211) × (100).

Table 2. Values of $U\begin{pmatrix}(20\ldots0) & (10\ldots0) & (\lambda)\\ (10\ldots0) & (20\ldots0) & (\mu)\end{pmatrix}$ for SO(2l+1).

	(λ)		
(μ)	$(10\ldots0)$	$(30\ldots0)$	$(210\ldots0)$
$(0\ldots0)$	$-\left(\frac{1}{l(2l+3)}\right)^{1/2}$	$\left(\frac{2l+5}{3(2l+3)}\right)^{1/2}$	$\left(\frac{2l-1}{3l}\right)^{1/2}$
$(20\ldots0)$	$\left(\frac{(2l-1)(2l+5)}{4l(2l+3)}\right)^{1/2}$	$-\left(\frac{2l-1}{3(2l+3)}\right)^{1/2}$	$\left(\frac{2l+5}{12l}\right)^{1/2}$
$(110\ldots0)$	$-\left(\frac{(2l+1)^2}{4l(2l+3)}\right)^{1/2}$	$-\left(\frac{2l+5}{3(2l+3)}\right)^{1/2}$	$\left(\frac{2l-1}{12l}\right)^{1/2}$

8. The group O(4)

When d is integral (and equal to l) an irrep $(w_1w_2\ldots w_l)$ of O(2l) gives rise to the two irreps $(w_1w_2\ldots\pm w_l)$ of SO(2l). To exemplify this, we set $2n=-4$ and $\sigma=2$ in the entries of table 1, thereby obtaining a set of U coefficients for O(4). They are displayed in table 3. In order to distinguish the irreps of O(4) from those of SO(4), the former are indicated by square brackets. The modification rules allow us to replace the heading [211] of the fifth column by [20], which we distinguish from the first column by adding the subscripts x and y. It is easy to confirm that [21]×[10] contains two irreps [20] from table D-1 of Wybourne (1970). The labels for the rows carry over directly from those given in table 1; however, [21]×[21] contains two irreps [00] and four irreps [20], so it may seem odd that only half of them appear in table 3. The reason for this stems from the fact that irreps of O(4) of the type $[w0]$ require an additional label (+ or −) to specify them uniquely (Elliott and Dawber 1984, vol 2, p 348). The appropriate choice for [00] and [20] depends on the corresponding choices that have implicitly been made already for the two irreps [10] appearing in the U coefficient. For example, if one is $[10]^+$ and the other $[10]^-$, their coupling to $[w0]$ forces this irrep to be of the type $[w0]^-$, and $[w0]^+$ plays no role. In this way the superfluous rows in table 3 disappear.

Since $SO(4)=SO_A(3)\times SO_B(3)$, where A and B distinguish the two components of the direct product, it should be possible to reconstruct table 3 from the U coefficients for SO(3). The only unknowns are the isoscalar factors involving the irreps of O(4)

Table 3. Values of $U\begin{pmatrix}[21] & [10] & [\lambda]\\ [10] & [21] & [\mu]\end{pmatrix}$ for O(4).

	$[\lambda]$				
$[\mu]$	$[20]_x$	[11]	[31]	[22]	$[20]_y$
[00]	3/8	$-(3/32)^{1/2}$	$(15/32)^{1/2}$	$-(5/32)^{1/2}$	3/8
$[20]_a$	$-(5/64)^{1/2}$	$-(15/32)^{1/2}$	$(3/32)^{1/2}$	$(9/32)^{1/2}$	$-(5/64)^{1/2}$
$[20]_b$	$-(1/2)^{1/2}$	0	0	0	$(1/2)^{1/2}$
$[11]_a$	$(1/8)^{1/2}$	$-(1/3)^{1/2}$	$-(5/12)^{1/2}$	0	$(1/8)^{1/2}$
$[11]_b$	$(5/32)^{1/2}$	$(5/48)^{1/2}$	$(1/48)^{1/2}$	3/4	$(5/32)^{1/2}$

and SO(4). It is not difficult to work out what they should be in order to make the correspondence between the U coefficients of O(4) and those of $SO_A(3)\times SO_B(3)$ complete. The connection is most readily expressed by writing

$$U\begin{pmatrix}[21] & [10] & [\lambda]\\ [10] & [21] & [\mu]\end{pmatrix}$$
$$=\langle(([10][21])[\lambda],[21],[10]|[10],([21][21])[\mu],[10]\rangle$$

and then expanding both the bra and the ket of the above recoupling coefficients in terms of their analogues in SO(4). For example,

$$\begin{aligned}&|[10],([21][21])[11]_a,[10]\rangle\\ &\quad=-(5/12)^{1/2}\{|(10),((21)(21))(11),(10)\rangle\\ &\qquad+|(10),((2-1)(2-1))(1-1),(10)\rangle\}\\ &\qquad-(1/12)^{1/2}\{|(10),((21)(21))(1-1),(10)\rangle\\ &\qquad+|(10),((2-1)(2-1))(11),(10)\rangle\}.\end{aligned}\tag{12}$$

The passage from SO(4) to $SO_A(3)\times SO_B(3)$ is accomplished by replacing an irrep (w_1w_2) of SO(4) by the product $j_A\times j_B$ of two angular momenta, where $j_A=\frac{1}{2}(w_1+w_2)$ and $j_B=\frac{1}{2}(w_1-w_2)$.

The coefficients in the expansion of equation (12) indicate that a non-trivial adjustment is required to find the 6-j symbols of SO(4) from those of O(4). When every irrep of O(2l) corresponds to a single irrep of SO(2l) (that is, when $w_l=0$), the isoscalars reduce to phase factors and the U coefficients of these two groups become essentially identical to one another. Of course, we could have calculated the 6-j symbols for SO(4) by using Cerkaski's method for this particular group directly, rather than going via Sp($-2d$). However, it is interesting to note that there is no provision in the formulation he gave of his method for making the extension from SO(4) to O(4). We should also point out that the appearance in our work of the 6-j symbols for O(4) rather than SO(4) could not have been anticipated from the analyses of Cvitanović and Kennedy (1982) and Dunne (1989), where the orthogonal-symplectic connection is discussed solely in terms of the duality Sp($-N$)$\leftrightarrow$SO(N).

9. Concluding remarks

The most striking feature of table 1 is that all the entries are the square roots of products of factors linear in σ. No quadratic forms appear. Not only that, but there are two apparently accidental zeros—possibly suggesting that there remains some structure still to be uncovered. Cerkaski's method has led to an elegant solution of the multiplicity problem for the particular example under study. Whether results of comparable simplicity occur in other cases or not is an open question. We were fortunate in that the relevant eigenvalues of Casimir's operator C_2 factorized into two parts with rational coefficients, This cannot be expected to happen very often, and any quadratic forms for which such a factorization cannot be made will necessarily be carried forward into the 6-j symbols themselves. However, the example we have chosen to study has worked out extremely nicely, and this augurs well for extensions to other cases.

As a final comment, we note that it is easy to pass from the U coefficients to 6-j symbols by dividing by the factor $[\mathrm{Dim}\langle\lambda\rangle\,\mathrm{Dim}\langle\mu\rangle]^{1/2}$, as mentioned in section 2. For special cases, the dimensions of the irreps may be found from the tables of Wybourne (1970) or those of McKay and Patera (1981). Of course, analytical formulae are readily available. The manipulations associated with the introduction of negative dimensions, which relate the Casimir operators for Sp($2n$) and O($2d$) when we set $n=-d$ (as described in section 5), can also be made in these formulae. This was noticed by King (1971) a long time ago. So the connections between the U coefficients of Sp($2n$) and O($2d$) go over immediately into connections between the corresponding 6-j symbols of these two groups.

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